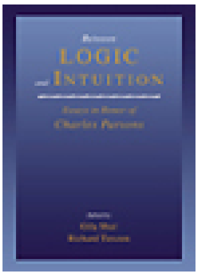

Between Logic and Intuition

Essays in Honor of Charles Parsons

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Chapter

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Geometry, Construction, and Intuition in Kant and His Successors

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I

I begin with an issue concerning the interpretation of the role of intuition in Kant's theory of geometry that has recently seen myself on one side and Charles Parsons on the other.[1] The interpretation I have defended is a version of what one might call the logical approach to Kantian intuition – an approach first articulated by Evert Beth and Jaakko Hintikka.[2] On this approach the primary role of Kantian intuition is formal or inferential: it serves to generate singular terms in the context of mathematical reasoning in inferences such as we would represent today by existential instantiation. Accordingly, the primary feature that distinguishes Kantian intuitions from purely conceptual representations, on this view, is their singularity – as opposed, that is, to the generality of concepts. Parsons has objected, however, that this formal–logical approach downplays a second feature that Kant also uses to distinguish intuitions from concepts: namely, their immediacy. For Kant, conceptual representation is both general and mediate, whereas intuitive representation is both singular and immediate – that is, it is immediately related to an object. And here, Kant certainly seems to think that the idea of immediacy adds something important – something of an epistemological and/or perceptual character – to the bare logical idea of singularity. Parsons himself suggests that the immediacy in question is to be understood as "direct, phenomenological presence to the mind, as in perception,"[3] and so, this second approach can be characterized as phenomenological. More specifically, the primary role of Kantian geometrical intuition, in this approach, is to acquaint us, as it were, with certain phenomenological or perceptual spatial facts, which can then be taken to provide us with evidence for or to verify the axioms of geometry. In this view, therefore, the question of the origin and justification of the axioms of geometry is prior to that concerning the nature and character of geometrical reasoning from these axioms, whereas on the logical approach, the priority is precisely the reverse.

The particular version of the logical approach to Kantian geometrical intuition that I have defended focuses on the role of Euclidean constructions in the proof procedure actually employed in the *Elements*, the procedure of construction with straight edge and compass articulated in Euclid's first three postulates. The idea is that all the objects introduced in Euclid's reasoning – points, lines, and so on – are iteratively or successively generated by straightedge and compass construction from a given line segment or pair of points. The existence of such objects – and, in particular, of an infinity of such objects – is not simply postulated, as in modern treatments, in existential axioms; rather, it is iteratively or successively generated from an initial object by given initial operations. In this sense, Euclid – again as opposed to modern treatments invoking Dedekind continuity, for example – is concerned only with constructive existence claims and with the potential infinite. Indeed, from this point of view, the existence of an infinity of geometrical objects appears precisely analogous to that of the natural numbers. Moreover, from this point of view, as I have further argued, we obtain a plausible explanation for why Kant thinks that geometrical representation is not purely conceptual. Conceptual representation, as Kant understands it, involves only the logical resources of traditional syllogistic logic. But, with these resources alone, we cannot represent an infinity of objects, not even the potential infinity of the natural numbers. Kant's recognition of this fact, together with his appreciation of the way in which Euclid himself represents the (potential) infinity of geometrical objects by a definite procedure of construction, can thus be taken to explain the Kantian doctrine that construction in pure intuition, and therefore geometrical space, is a nonconceptual species of representation.

On this interpretation, then, the infinity of space is a purely formal–logical feature of mathematical geometry (we would express it today by saying that formal systems of geometry have only infinite models), and the intuitive, nonconceptual character of the representation of space is a consequence of this same formal–logical feature (we would express it today by saying that *monadic* formal systems always have finite models if they have models at all). So, phenomenological or perceptual features of our representation of space play no role whatever here. By contrast, as has been brought out especially clearly in a recent paper by Emily Carson,[4] on the phenomenological approach favored by Parsons the order of explanation is precisely the reverse. The infinity of space is a directly given perceptual fact – it consists of the circumstance that any perceived spatial region belongs within a larger "horizon" as part of a single, uniquely given perceptual space[5] – and it is this perceptual fact that then justifies or explains the use of infinity in mathematical geometry. Perceptual space supplies the framework, we might say, within which geometrical construction takes place and which, accordingly, guarantees that the constructions postulated by Euclid can indeed by carried out.[6] Even if Kant were acquainted with modern, purely logical formulations of mathematical geometry, therefore, he would still need to appeal to spatial intuition – that is, to phenomenological features of our spatial perception – to justify or to verify the relevant axioms.

Now there is an important Kantian text that bears decisively on this issue. It belongs to the dispute with Eberhard in 1790 and occurs in Kant's handwritten notes that were used almost verbatim by his disciple Johann Schulze in the latter's review of Eberhard's *Philosophisches Magazin*. In particular, the passage in question was used by Schulze in his review of essays by the mathematician Abraham Kästner, and it shows, I believe, that the logical approach to Kantian geometrical intuition must, at the very least, be supplemented by considerations congenial to the phenomenological approach. Kant begins by distinguishing space as described by geometry from space as described by metaphysics. The former is generated [*gemacht*] or derivative, and, in this sense, there are many *spaces*. The latter is given or original, and, in this sense, there is only one single *space*. Kant leaves no doubt, moreover, that the infinity of geometrical spaces is grounded in the single, uniquely given metaphysical space:

[T]he representation of space (together with that of time) has a peculiarity found in no other concept, viz., that all spaces are only *possible* and thinkable as parts of one single space, so that the representation of the parts already presupposes the representation of the whole. Now, when the geometer says that a line, no matter how far it has been extended, can still always be extended further, this does not mean the same as what is said in arithmetic concerning numbers, viz., that they can be always and endlessly increased through the addition of other units or numbers (for the added numbers and magnitudes that are expressed thereby are possible in themselves, without needing to belong together with the previous ones as parts of a whole). Rather, to say that a line can be continued to infinity means that the space in which I describe the line is greater than any line that I may describe in it. Thus, the geometer grounds the possibility of his problem – to increase a space (of which there are many) to infinity – on the original representation of a single, infinite, *subjectively given* space. This agrees very well with the fact that geometrical and objectively given space is always *finite*. For the latter is only given in so far as it is generated [*gemacht*]. To say, however, that the metaphysical, i.e., original but merely subjectively given space – which (because there are not many) cannot be brought under any concept capable of construction but which still contains the ground of all possible constructions – is *infinite* means only that it consists of the pure form of the mode of sensible representation of the subject as a priori intuition. Therefore, the possibility of all spaces, which proceeds to infinity, is *given* in this space as a singular representation. (Ak. 20, pp. 419–21)[7]

The explicit contrast that Kant draws here between the infinity of space and that of the natural numbers is made even sharper a few lines later when Kant endorses the idea that mathematics considers only the potential infinite [*infinito potentiali*] and that "an infinity in act [*actu infinitum*] (the metaphysically-given) is not given on the side of the object, but on the side of the thinker" – where the latter "infinity in act" nevertheless "lies at the basis of the progression to infinity of geometrical concepts."

It is clear, therefore, that the analogy between the infinity of space and that of the natural numbers can only apply to space as described by the geometer.

Geometry deals with a successively generated sequence of spaces (spatial objects) that is potentially infinite as a whole and thus necessarily finite at every stage. By contrast, space as described by the metaphysician – as described in the Metaphysical Exposition of the Concept of Space in the Transcendental Aesthetic of the first *Critique* – has quite a different character. And it is precisely this special character of metaphysical space, moreover, which, for Kant, grounds or explains the possibility of the infinity of space as described by the geometer. So far, then, the basic ideas of the phenomenological approach appear to be vindicated. The crucial question, however, concerns exactly how metaphysical space – "the pure form of the mode of sensible representation of the subject as a priori intuition" – is supposed to accomplish this grounding. Is the given infinity of space as a pure form of sensible intuition supposed to be directly seen, as it were, in a simple act of perceptual or quasi-perceptual acquaintance? Are we supposed to have direct perceptual or quasi-perceptual access to such infinity entirely independently of geometry – which access we can then use to justify or to verify the possibility of Euclidean constructions? Both of these ideas appear to be very doubtful. For we are certainly not perceptually presented with an infinite space as a single given whole; and, since the visual field is itself always finite, it does not even appear to be true that any perceived spatial region is directly given or perceived as part of a larger such region. The idea of independently given phenomenological facts capable of somehow grounding or justifying the possibility of geometrical construction can quickly appear to be absurd.

Several pages earlier in the same notes for Schulze's reply to Kästner, Kant himself discusses the question of explaining or justifying the possibility of geometrical construction as follows:

> [I]t is very correctly said [by Kästner] that "Euclid assumes the possibility of drawing a straight line and describing a circle without proving it" – which means without proving this possibility *through inferences*. For *description*, which takes place a priori through the imagination in accordance with a rule and is called construction, is itself the proof of the possibility of the object. Mechanical *delineation* [*Zeichnung*], which presupposes description as its model, does not come under consideration here at all. However, that the possibility of a straight line and a circle can be proved, not *mediately* through inferences, but only immediately through the construction of these concepts (which is in no way empirical), is due to the circumstance that among all constructions (presentations determined in accordance with a rule in a priori intuition) some must still be *the first* – namely, the *drawing* [*Ziehen*] or describing (in thought) of a straight line and the *rotating* of such a line around a fixed point – where the latter cannot be derived from the former, nor can it be derived from any other construction of the concept of a magnitude. (Ak. 20, pp. 410–11).

What grounds or explains the possibility of geometrical construction, then, is simply the immediate activity of our a priori imagination by which we draw or describe a straight line in thought and then rotate such a line around a fixed point. Indeed, Kant had already made it clear in the first *Critique* that it is precisely such imaginative activity that grounds the axioms of geometry:

I can represent no line to myself, no matter how small, without drawing it in thought, that is gradually generating all its parts from a point, and thereby first registering this intuition.... On this successive synthesis of the productive imagination in the generation of figures is based the mathematics of extension (geometry), together with its axioms, which express the conditions of a priori sensible intuition under which alone the schema of a pure concept of outer appearance can arise. (A162–3/B203–4)

And, as this passage intimates, the axioms of geometry are capable of no further proof, because it is only via the imaginative activity in question that the relevant geometrical concepts can be thought (compare A234–5/B287).

This last idea is given special emphasis in §24 of the second edition Transcendental Deduction, which also further articulates the imaginative activity in question. After characterizing the activity of the productive imagination as "figurative synthesis" or "transcendental synthesis of the imagination," and explaining that figurative synthesis is a "transcendental action of the imagination" expressing the "synthetic influence of the understanding on inner sense," Kant illustrates his meaning as follows:

We always observe this in ourselves. We can think no line without *drawing* it in thought, no circle without *describing* it. We can in no way represent the three dimensions of space without *setting* three lines at right angles to one another from the same point. And we cannot represent time itself without attending, in the *drawing* of a straight line (which is to be the outer figurative representation of time), merely to the action of synthesis of the manifold, through which we successively determine inner sense, and thereby attend to the succession of this determination in it. Motion, as action of the subject (not as determination of an object*), and thus the synthesis of the manifold in space – when we abstract from the latter and attend merely to the action by which we determine *inner* sense in accordance with its form – [such motion] even first produces the concept of succession. (B154–5)

And in the footnote, Kant explicitly links motion in the relevant sense with the imaginative description of space underlying the axioms of geometry:

*Motion of an *object* in space does not belong in a pure science and thus not in geometry. For, that something is movable cannot be cognized a priori but only through experience. But motion, as the *describing* of a space, is a pure act of successive synthesis of the manifold in outer intuition in general through the productive imagination, and it belongs not only to geometry, but even to transcendental philosophy.

Thus motion in the relevant sense – the pure act of successive synthesis in space as transcendental activity of the subject – grounds or underlies geometry by also belonging to the metaphysical consideration of space characteristic of transcendental philosophy. As Kant puts it in §26 in the footnote at B161, it is "through it [i.e., the transcendental synthesis of the imagination] (in that the understanding determines sensibility) that space or time are first *given*" as intuitions.

In what sense is the motion in question an "action of the subject"? Kant states in §24 that the understanding, as active subject, exerts the transcendental synthesis of the imagination on the "passive subject [i.e., inner sense] whose faculty it is" (B153). But it is also the case, as Kant explains at the very beginning of the Metaphysical Exposition of the Concept of Space, that the subject of outer sense is itself in space. Space as the form of outer sense enables us to represent objects as outer precisely by representing them as spatially external to the perceiving subject:

> Space is no empirical concept that has been derived from outer experiences. For, in order that certain sensations are related to something outside me (that is, to something in another place in space than the one in which I find myself), and, similarly, in order that I be able to represent them as outside of and next to one another – and thus not merely as different but as in different places – the representation of space must already lie at the basis. Therefore, the representation of space cannot be obtained from the relations of outer appearance through experience; rather, this outer experience is itself only possible in the first place by means of the representation in question. (A23/B38)

Space as the form of outer sense contains the point of view of the subject, from which the objects of outer sense are perceived and around which, as it were, the objects of outer sense are arranged.[8] And it follows that by changing this point of view – by moving in and through space – the subject can change its perspective on the objects of outer sense.

Let us suppose, then, that by the fundamental transcendental action of the imagination that Kant calls "figurative synthesis," the subject imaginatively locates itself in space at a definite position (a particular point of view) and with a definite orientation (a particular perspective on the spatial world as perceived from this point of view). Such an orientation is established, for example, by choosing three particular line segments set at right angles from a common point.[9] The objects of outer sense then appear as arranged around this subjective point of view and thus capable of being seen from it in accordance with the chosen orientation or perspective. And this much, moreover, belongs to the a priori structure of *pure* spatial intuition. In particular, *empirical* spatial intuition or perception is necessarily conceived as taking place within this already-established formal structure. Empirical spatial intuition occurs when an object spatially external to the point of view of the subject affects this subject – along a spatial line of sight, as it were – so as to produce a corresponding sensation in it; and it is in this sense, therefore, that the pure form of (spatial) sensible intuition expresses the manner in which we are affected by (outer) objects. It is in this sense, too, I believe, that the immediacy of pure spatial intuition, in contrast to the mediate character of merely conceptual representation, is to be understood. For, by virtue of the formal or a priori structure of spatial perception, pure spatial intuition contains causal, indexical, and demonstrative elements not present in merely conceptual representation. Pure spatial intuition thereby expresses the a priori form, we might say, by which the perceiving subject can be immediately related to (outer) objects.[10]

By the same fundamental action of the transcendental imagination through which the subject imaginatively locates itself at a given point of view with a given orientation, this subject can also, as noted above, imaginatively change the given point of view and orientation by imaginatively moving in and through space. In particular, the subject can imaginatively change the given point of location by a translation through space and imaginatively change its given orientation by a rotation around this point. Moreover, by an appropriate combination of such translations and rotations the subject can thereby imaginatively put itself in position to perceive *any* outer object located *anywhere* in perceptual space. It is in this sense, I believe, that perceptual space is necessarily both singular or unitary and infinite or unbounded. Perceptual space is singular or unitary because any outer object must be perceivable by the same perceiving subject, and thus all outer objects must be located within the same formal structure of possible perceptual relations: all outer objects must be reachable via translation and rotation, as it were, from a single initial given point of view. By the same token, perceptual space is infinite or unbounded because, although any particular momentary visual field is indeed bounded or finite, by moving in and through space and thereby changing its perspective, the subject changes its visual field so as to embrace successively more and more regions of the single, unitary perceptual space. It is in this sense – that is, kinematically – that any given spatial region is perceived within an "horizon" eventually comprising all possible perceptual spatial objects. And this also clarifies the sense, it seems to me, of the otherwise puzzling idea that metaphysical space – that is, the formal structure of perceptual space described in the Metaphysical Exposition – involves an "infinity in act [*actu infinitum*] (the metaphysically-given) [that] is not given on the side of the object, but on the side of the thinker."

Finally, this same formal structure of perceptual space can be seen as grounding or explaining the constructive procedure expressed in the axioms of Euclidean geometry. As we have seen, Kant takes the possibility of straight-edge and compass construction to be grounded in the imaginative activity of drawing a straight line via the rectilinear motion of a given point (here, in connection with B154–5, see especially B292) and describing a circle via the rotation of a given line. The construction of a straight line, in other words, is executed by a translation, and the construction of a circle is executed by a rotation. (In modern terminology, straight lines and circles appear as orbits of the Euclidean group of motions.) The possibility of translational and rotational motion is primary, therefore, because it is given in the pure formal structure of perceptual space. Geometrical space is then iteratively or constructively generated within the formal structure of perceptual space by successively applying the fundamental operations of drawing a straight line and describing a circle, and this is the precise sense, I believe, in which the possibility of mathematical geometry is grounded in or explained by the formal structure of perceptual space.[11] It does not necessarily follow, however, that the structure of perceptual space can be taken to provide an independent epistemological justification for the axioms of geometry – still less that there is an independently accessible realm of phenomenological facts capable of providing such a justification through some kind of quasi-perceptual direct acquaintance. The spatial intuition grounding the axioms of geometry is fundamentally kinematical, in my view, and it is expressed in the formal structure of translational and rotational motion (in modern terminology, the structure of the group of Euclidean motions). That perceptual space in fact has or embodies this formal structure can in no way be simply read off of our perceptual experience, as it were, independently of our knowledge of geometry. On the contrary, the only way in which we know that perceptual space in fact embodies this structure is precisely through our knowledge that geometry is applicable to it.[12] Kant's theory of pure spatial intuition provides an explanation of the possibility of geometry – an explanation, in particular, of the nonconceptual and intuitive or perceptual character of geometry. But it does not provide, and does not attempt to provide, an independent epistemological foundation.[13]

II

Giving motion – understood in terms of the possible changes in position and orientation of the subject's point of view – a central role in Kant's doctrine of pure spatial intuition raises a variety of interpretative questions concerning, on the one hand, the scope of geometrical construction and, on the other, the involvement of the understanding and the transcendental unity of apperception in the characteristic features of sensibility.

With respect to the first set of questions, it is noteworthy that Kant himself is forced explicitly to reconsider the nature and scope of geometrical construction in the course of the dispute with Eberhard – during the very same period, therefore, when our initial texts on the infinity of space were written. Eberhard had appealed to Apollonius's treatise on conic sections in order to argue against the Kantian doctrine that geometrical concepts require the construction of a corresponding intuition. For Apollonius uses the defining planar characterization of a parabola, $y^2 = ax$, without showing how to draw or delineate such a curve in the plane. Indeed, although every individual point on the curve can be constructed by straight edge and compass, the curve itself cannot be continuously traced thereby; the latter task requires more complicated means of construction, which are often, in contradistinction to "geometrical" constructions with straight edge and compass, termed "mechanical" constructions. Yet, since Apollonius is able to develop the entire theory of conic sections without considering such mechanical constructions at all, it cannot be true that the successful use of geometrical concepts always requires the construction of a corresponding intuition.[14]

Kant's reply is twofold. He objects, in the first place, that Apollonius does indeed provide a construction in intuition – not, to be sure, in the plane but rather in space:

Apollonius first constructs the concept of a cone, that is, he presents it a priori in intuition (this is now the first action whereby the geometer verifies beforehand the objective reality of his concept). He cuts it in accordance with a determinate rule, e.g., parallel to a side of the triangle that cuts the base of the cone (*conus rectus*) through its apex at right angles, and proves a priori in intuition the properties of the curved line that is generated by means of this section on the surface of the cone. He thus brings forth a concept of the ratio in which the ordinates of this curve stand to the diameter [i.e., the relation $y^2 = ax$], which concept, namely (in this case) of the parabola, is thereby given a priori in intuition; and therefore its objective reality – that is, the possibility that a thing with the properties in question can be given – is proven in no other way *except that one supports it with the corresponding intuition*. (Ak. 8, p. 191)

Thus, in his very first definition, Apollonius generates a cone by rotating an infinite straight line in space, fixed at a given point, around the diameter of a given circle (the base of the cone). In Proposition I.11, he generates the parabola from a section whose diameter is parallel to one of the sides of the axial triangle, and proves thereby that the characteristic equation, $y^2 = ax$, where a is the so-called latus rectum or parameter, then holds.[15] The curve is thus derived or constructed in space, and for this reason, the ancients termed problems involving conic sections, in contradistinction to "plane" problems constructible by straight edge and compass, "solid" problems.[16]

Kant further objects, in the second place, that mechanical constructions in the plane are completely irrelevant:

[Apollonius's editor, Borelli[17]] speaks of mechanical *construction* of the concepts of the conic sections (except for the circle) and says that mathematicians teach the properties of the latter without mentioning the former – which is certainly a true observation but a very insignificant one; for instructions on how to *delineate* [*zeichnen*] a parabola in accordance with the prescriptions of the theory are only for the artist, not for the geometer.*

And the footnote then clarifies the distinction that Kant has in mind here:

*The following may serve to secure against the misuse of the expression, the *construction* of concepts, of which the *Critique of Pure Reason* frequently speaks and by which it has first precisely distinguished the procedure of reason in mathematics from its procedure in philosophy. In the more general meaning, all *presentation* of a concept through the (spontaneous) production of an intuition corresponding to it can be called construction.

If this takes place through the mere imagination in accordance with an a priori concept, it is called *pure* construction (which the mathematician must lay at the basis of all his demonstrations...). If, however, it is exerted on any kind of matter, it could be called *empirical* construction. The former can also be called schematic, the latter technical. The latter type of construction, which is actually only improperly so-called (because it belongs not to science but to art and is achieved with instruments), is now either *geometrical* construction by means of compass and ruler or *mechanical* construction, for which other instruments are necessary – as, for example, the delineation [*Zeichnung*] of the remaining conic sections besides the circle. (Ak. 8, pp. 191–2)

In particular, then, Kant himself is perfectly aware that the general conic section is not constructible with straight edge and compass.

In this same passage, however, Kant classifies ruler and compass constructions as "empirical" or "technical" as well. Does this mean that even the constructions of elementary Euclidean geometry are also mathematically irrelevant? That this is emphatically not the case is clear from a footnote to §I of the *First Introduction to the Critique of Judgement* (which is thus written shortly before the reply to Eberhard):

*This pure and for precisely this reason noble science [i.e., geometry] seems to compromise its dignity when it admits that, as elementary geometry, it uses *instruments*, although only two, for the construction of its concepts – namely the compass and the ruler, which constructions alone it calls geometrical, while those of higher geometry, by contrast, it calls mechanical since for the construction of the concepts of the latter more composite [*zusammengesetztere*] machines are required. But one also understands by the former, not the actual instruments (*circinus et regula*), which could never give these figures with mathematical precision; rather, they should mean only the simplest modes of presentation of the a priori imagination, which no instrument can imitate. (Ak. 20, p. 198)

And it is clear from this passage, together with the passage on the transcendental synthesis of the imagination at B154–5 and the passage from the reply to Kästner at Ak. 20, pp. 410–11 cited earlier, that the "simplest modes of presentation of the a priori imagination" in question are just the two activities (in thought) of drawing [*ziehen*] a straight line and describing a circle (by rotating a line segment about a fixed point in a plane). Construction of straight lines and circles, when performed via figurative synthesis in the a priori imagination rather than with real draftsman's instruments on real pieces of paper, thus continues to be paradigmatic of "properly so-called" pure or schematic mathematical construction.

There are therefore two different distinctions at play here. The first is a distinction, within mathematics, between those curves or figures constructible via straight lines and circles and more complex figures such as the conic sections. The former are constructible by straight edge and compass in the idealized mathematical sense, and this question is entirely independent of the capabilities of any actual draftsman's instruments. The second distinction, by contrast, is between pure mathematical construction, whether belonging to "elementary" or to "higher" geometry, and actual empirical delineation. In this sense, even constructions with straight edge and compass, when it is a matter of actual draftsman's instruments rather than idealized mathematical operations, lie outside the concerns of the geometer. Kant's dichotomy between schematic and technical construction precisely corresponds to the second distinction, but his use of "mechanical construction" tends to blur the two. In the passage from the reply to Kästner at Ak. 20, pp. 410–11, for example, a priori mathematical "description" is opposed to "mechanical delineation," and a few lines later in the above reply to Eberhard Kant sets up an opposition between "pure, merely schematic construction" and "*mechanical* [construction]" (Ak. 8, 192). This latter usage of "mechanical" thus corresponds to "empirical" or "technical" and does not involve the contrast between ruler and compass and more complicated forms of construction.[18]

Lying behind this ambiguity is an important issue of principle concerning the precise scope of admissible geometrical or mathematical operations – an issue made particularly acute by the investigation of a large variety of new curves in the seventeenth and eighteenth centuries. The *locus classicus* for this issue is Descartes's *Géométrie*, which, as is well known, develops a novel version of the distinction between "geometrical" and "mechanical" curves. The former include all algebraic curves – not only lines, circles, and the conic sections but also curves defined by algebraic equations of third and higher degree – whereas the latter comprise the nonalgebraic or transcendental curves. Moreover, the algebraic curves, according to Descartes, are all constructible by appropriate generalizations of the straight edge and compass, by idealized instruments that arise by iteration, as it were, of the most elementary ones. We can construct lines and circles and then rotate and translate them to produce new curves (like the conic sections); we can then rotate and translate these new curves to produce further curves (like the so-called Cartesian parabola, which is of the third degree); and so on. Transcendental curves, by contrast, do not find a place in this iterative hierarchy of possible constructions. They may exist "mechanically" in actually given empirical nature, but they forever exceed our precise mathematical grasp – our capacity clearly and distinctly to proceed step-by-step via intuitively evident rules.[19]

Unfortunately, there is not enough evidence to determine where Kant himself stands on this issue. He never, to my knowledge, considers curves more complex than the conic sections, and here his viewpoint appears to be entirely traditional. Conic sections are intuitively presentable through the ancient "solid" constructions on a cone, which itself arises through the rotation of a line with a fixed point in space. For Kant, as we have seen, what is primary are the basic operations – the "simplest modes of presentation of the a priori imagination" – by which the subject can execute translations of and rotations around a given point of view in space, and he appears to hold that only constructions that can arise thereby are geometrically and mathematically admissible. Yet some delimitation of what "can arise thereby" actually means is necessary if a notion of admissible construction is to be at all well defined, and Kant unfortunately says nothing to suggest such a delimitation. If arbitrary combinations of translations and rotations are allowed, we can clearly construct any continuous curve, and then there is no reason, in particular, to dismiss "mechanical" constructions of the conic sections as mathematically irrelevant.[20] So what is needed, then, is some iterative extension of a set of basic operations analogous to Descartes's. From a fundamental logical and mathematical point of view, however, the issue proves to be a deep one indeed. For it eventually leads, via the need to assimilate transcendental as well as algebraic curves, to the free use of infinitesimal methods and thus, in the end, to the realization that a radically new type of iteration essentially involving limit operations is required.

In any case, the relationship that Kant does set up between geometrical construction, on the one hand, and motion in space (i.e., translations and rotations), on the other, raises significant questions about his own doctrine of sensibility. In particular, if, as we argued above, the two key features of intuitive space – its unity and infinity – depend on the possible motions of the subject's point of view, then these features appear to depend on the unity and identity of the subject – and thus, in the end, on the transcendental unity of apperception – rather than on independently given features of space as a form of sensibility. Space is unitary because every possible object therein must be reachable from a given initial point of view by an appropriate combination of translations and rotations; and space is infinite or unbounded because any initial perceptible region is thereby extendible without limit to any other perceptible region. These two key features of intuitive space therefore directly depend on the requirement that every spatial region be accessible via continuous motion by a single perceiving subject, and without this requirement there would simply be no guarantee whatever that all possible spatial regions belong to a single, unitary and unbounded, comprehensive system of such regions.[21]

This dependence of key features of sensibility on the transcendental unity of apperception, and thus on the understanding, is closely related, in turn, to the well-known distinction that Kant makes between space as "form of intuition" and as "formal intuition" in §26 of the second edition Transcendental Deduction:

*Space represented as *object* (as is actually required in geometry) contains more than the mere form of intuition – namely, [it contains] *uniting* [*Zusammenfassung*] of the manifold in accordance with the given form of sensibility in an *intuitive* representation, so that the *form of intuition* gives [a] mere manifold but the *formal intuition* gives unity of representation. In the Aesthetic I counted this unity [as belonging] to sensibility, only in order to remark that it precedes all concepts, although it in fact presupposes a synthesis that does not belong to the senses but through which all concepts of space and time first become possible. For, since through it (in that the understanding determines sensibility) space or time are first *given*, the unity of this a priori intuition belongs to space and time, and not to the concept of the understanding (§24). (B160–1)

The reference to geometry and to §24 implies, I believe, the conception of motion in space that was first suggested above by the passage at B154–5. And that geometrical motion in this sense is a direct expression of the transcendental unity of apperception is explicitly stated in §17:

Therefore, the first pure cognition of the understanding, on which its entire remaining use is grounded, and which is also, at the same time, entirely independent of all conditions of sensible intuition, is the principle of the original *synthetic* unity of apperception. Thus, the mere form of outer sensible intuition, space, is not yet any cognition at all; it gives only the manifold of a priori intuition for a possible cognition. But to cognize anything in space, for example, a line, I must *draw* [*ziehen*] it and therefore bring about synthetically a determinate combination of the given manifold – so that the unity of this action is, at the same time, the unity of consciousness (in the concept of a line), and thereby alone is an object (a determinate space) first cognized. (B137–8)

The unity of the intuitive representation in question therefore depends directly on the unity of consciousness and thus, in the end, on a conceptual unity.

Why, then, does Kant also assert, in the last sentence of the footnote at B160–1, that "the unity of this a priori intuition belongs to space and time, and not to the concept of the understanding"? The point, I think, is that the relationship between the understanding and sensibility effected by the transcendental synthesis of the imagination is a reciprocal one. To be sure, space would not be unitary in the relevant sense without the "action of the understanding on sensibility" (B152) manifested in figurative synthesis. Nevertheless, the unity thereby produced is not itself a conceptual unity, whereby a number of representations (subordinate concepts) are contained *under* a given representation (superordinate concept); it is, rather, a distinctly intuitive unity, whereby a number of representations (spatial regions) are contained *in* a given representation (that of a single space) (see B39–40 and compare B133 n). All spatial regions belong to a single space in that they all must be reachable from here, as it were, but reachable-from-here is not a conceptual relation. By the same token, although the transcendental synthesis of the imagination is a realization or embodiment of the "pure intellectual synthesis" contained in the synthetic unity of apperception (B150–2), it must also go beyond pure intellectual synthesis since the latter in fact requires pure sensibility if it is to succeed in unifying a given manifold (B153–4). There would thus be no unity in the relevant sense without the mutual cooperation of understanding and sensibility, without that interaction between the two faculties "through which the categories, as mere forms of thought, then acquire objective reality, that is, application to objects that can be given to us in intuition" (B150–1).

Even in pure mathematical synthesis in the pure imagination, therefore, the categories are necessary to bring unity into the intuitive manifold – a point Kant makes explicitly in §20 of the *Prolegomena*:

> Even the judgements of pure mathematics in its simplest axioms are not exempt from this requirement [i.e., subsumption under a pure concept of the understanding]. The principle that the straight line is the shortest between two points presupposes that the line is subsumed under the concept of magnitude, which is certainly no mere intuition but has its seat solely in the understanding and serves to determine the intuition (the line) with respect to the judgements that may be made of it in relation to their quantity, namely [in relation to] plurality ... (Ak. 4, pp. 301–2)

And in the Table of Pure Concepts of the Understanding following in §21, Kant lists the categories of quantity in the form: "unity (the measure), plurality (the magnitude), totality (the whole)" (p. 303). So, it is clear, then, that the categories involved in pure mathematical synthesis, and thus in the mathematical unity of space as an object of geometry, are the categories of quantity.[22]

In the terminology of the important footnote added at B201, we are therefore involved with a *mathematical* synthesis of "*composition* (*compositio*)" rather than a *dynamical* synthesis of "*connection* (*nexus*)." We are involved with the mathematical categories (here the categories of quantity) rather than the dynamical categories of relation and modality – where the former ground the possibility of the principles of mathematics, whereas the latter ground the possibility of "general (physical) dynamics" (A162/B202). And this distinction has fundamental implications for the nature and status of the pure geometrical motion (viz., translation and rotation), which, according to our interpretation, first embodies mathematical synthesis. In particular, since the dynamical categories and thus "general (physical) dynamics" are not yet at issue, we are not concerned here with the questions about distinguishing true from apparent motion and establishing a privileged frame of reference arising in the context of Newtonian dynamics.[23] The motion with which we are concerned here is purely relative or, perhaps better, purely mathematical, in that we abstract from all questions of speed, acceleration, duration, and so on, and attend only to its character as a continuous transformation.[24]

III

The above interpretation of Kantian spatial intuition attempts to build a bridge between the phenomenological and logical approaches by viewing the relevant formal structure of intuitive or perceptual space as fundamentally kinematical: it is a matter of the possible translational and rotational motions (in modern terms, the group of rigid motions) by which the perceiving subject can move in and through space so as to put itself in potential perceptual contact with all possible spatial objects. This structure then grounds the formal procedure of geometrical construction underlying pure mathematical geometry by generating the two basic operations of drawing straight lines and describing circles (in modern terms, as orbits of the group of motions). From this point of view, therefore, Kant's own conception of spatial intuition is not so far from that developed in the nineteenth century by Hermann von Helmholtz. Indeed, it is of course Helmholtz who first explicitly articulates a program for founding geometry on the formal structure of perceptual space based, via the condition he calls free mobility, on the group of rigid motions.[25] Nevertheless, it is well known that Helmholtz presents his position as anti-Kantian, and this in two central respects.

The first and most obvious respect in which Helmholtz presents his conception as anti-Kantian is that Helmholtz explicitly attacks the idea that the specific structure of *Euclidean* space is grounded in our spatial intuition or is in any way a priori. From Riemann's work and his own mathematical investigation of the "space-problem"[26] Helmholtz has learned that the relevant formal structure of possible motions in and through perceptual space (the structure characterized by the condition of free mobility) does not yield specifically Euclidean space, but rather the three classical cases of spaces of constant curvature: Euclidean space, spherical or elliptic space, and hyperbolic space. By imagining a mobile perceiver located in one or another of the non-Euclidean spaces of constant curvature, we can then make it perfectly evident that the formal structure of spatial intuition alone (i.e., the possibility of free mobility) does not uniquely single out the Euclidean case:

> This will suffice to show how one can, in the way suggested, derive from the known laws of our sensible perceptions the series of sensible impressions that a spherical or pseudo-spherical world would give us if such a world existed. We thereby never come upon an inconsistency or impossibility, any more than in the calculative treatment of metrical relationships. We can picture to ourselves the appearance of a pseudo-spherical world outwards in all directions, just as well as we can develop the concept of such a world. We therefore cannot grant that the axioms of our geometry [i.e., Euclidean geometry] are grounded in the given form of our faculty of intuition or are somehow implicated in such a form.[27]

And it is in this sense, therefore, that Helmholtz defends an empiricist conception of geometry. The axioms of specifically Euclidean geometry are neither necessities of thought (because we can consistently develop the more general concept of Riemannian metrical manifold) nor necessities of intuition (because the formal structure of spatial perception leaves all three classical cases of constant curvature still open). Specifically Euclidean geometry thus can be obtained only from the observed facts governing the behavior of mobile rigid bodies in the actual world. (Helmholtz himself has no reason to doubt, of course, that the observed facts do indeed support specifically Euclidean geometry.)

This does not mean, however, that the Kantian idea of spatial intuition and its a priori structure are wholly erroneous. On the contrary, Helmholtz's famous assertion that "space can be transcendental without the axioms being so" is intended precisely to underscore the fundamental truth that he finds in the Kantian doctrine:

Kant's doctrine of the a priori given forms of intuition is a very happy and clear expression of the situation. Yet this form must be contentless and free enough in order to take up every content that can ever enter into the form of perception in question. But the axioms of geometry [i.e., Euclidean geometry] limit the form of intuition of space so that every thinkable content can no longer be taken up therein, if geometry is to be at all applicable to the actual world. However, if we leave these axioms aside, then the doctrine that the form of intuition of space is transcendental is free from any stumbling block. Here Kant was not critical enough in his critique. Certainly, however, we are here concerned with propositions of mathematics, and this piece of the critical work must be taken care of by the mathematician.[28]

Since our spatial intuition in fact has an a priori formal structure expressed mathematically in the condition of free mobility, Kant's doctrine is, so far, unobjectionable. Only the later mathematical discovery of the classical non-Euclidean geometries and the fact that precisely the *three* classical cases are given by the condition of free mobility allow us to correct the one flaw in Kant's original doctrine (which discoveries, we might add, Kant could in no way have been expected to anticipate).[29]

The second, and, in the present context, perhaps even more interesting respect in which Helmholtz presents his conception as anti-Kantian concerns the idea of spatial intuition itself. For Helmholtz presents his kinematical picture of spatial intuition expressed in the condition of free mobility – the picture of spatial intuition as involving the formal structure of the possible changes in point of view and orientation of the perceiving subject – as explicitly opposed to the "popular" or "older" concept of intuition (which he sometimes attributes to Kant himself, but sometimes only to the "Kantians of strict observance") according to which spatial intuition is a simple and unanalyzable momentary psychological act providing us with direct "evidence in a flash [*blitzähnliche Evidenz*]." And it is only by invoking what he takes to be his new, kinematical picture of spatial intuition that Helmholtz is then able to argue against the claims of contemporary Kantians that, although non-Euclidean geometries may be mathematically thinkable, they are not spatially intuitable and therefore are not possible candidates for describing the structure of our spatial intuition.[30] If the above interpretation of Kant's own doctrine of spatial intuition is at all correct, however, it turns out that at least the germ of Helmholtz's kinematical conception is already present in Kant himself. In this sense, Kant's explanation of the a priori status of Euclidean geometry in terms of the necessary structure of our pure form of spatial intuition contains the seeds of its own destruction.

Here, however, it is imperative to note a third respect in which Helmholtz's conception is very definitely anti-Kantian – a central difference between the two conceptions of spatial intuition that Helmholtz, because of his naturalistic transformation of the meanings of "a priori" and "transcendental," does not himself emphasize at all. For what Helmholtz considers as belonging to the a priori or transcendental structure of spatial intuition involves, from a Kantian perspective, *empirical* rather than *pure* intuition. Helmholtz constructs the relevant group of rigid motions expressing the free mobility of the perceiver in and through perceptual space from the muscular and kinesthetic sensations of the subject as it voluntarily initiates such motions, which motions are essentially considered, therefore, as movements of the subject's body. The sense in which the structure of these bodily sensations constitutes an a priori or transcendental form of intuition, then, is simply that this structure belongs on the side of the subject and does not simply picture or mirror an external realm of "things in themselves."[31] For Kant, by contrast, the relevant group-theoretical structure involves only the motions of a disembodied point of view and has nothing to do, therefore, with any bodily sensations. Kant is concerned only with that "action of the understanding on sensibility" (B152) whereby the (transcendental) subject locates itself in space at a definite point of view and with a definite orientation.[32] Indeed, Kantian pure, as opposed to empirical, intuition can, of course, involve no sensations or actual perceptions at all.[33] Kant's doctrine of space as a *pure* form of outer intuition is in this sense entirely unique, and it cannot be satisfactorily understood, I believe, except by invoking the basic ideas of the logical interpretation of this doctrine. It is only because there is no room in Kant's own conception of logical, conceptual, or analytic thought for anything corresponding to pure mathematical geometry that there is a place, accordingly, for a wholly nonconceptual faculty of pure spatial intuition. For Helmholtz, by contrast, there is no difficulty at all in formulating pure mathematical geometry conceptually or analytically with no reference to spatial intuition whatsoever (via the Riemannian conception of metrical manifold), and an appeal to spatial intuition or perception is only then necessary to explain the psychological origin and empirical application of the pure mathematical concept of space.[34]

This necessary separation, in Helmholtz's conception, of the pure mathematical concept of space from perceptual or intuitive space implies, moreover, that there is a fundamental gap between the precision and exactitude of the mathematical concept and the rough and approximate character of the relevant perceptual or intuitive experience. And it is precisely by emphasizing and exploiting this gap that Henri Poincaré develops his contrasting conventionalist interpretation of geometry. For Poincaré entirely agrees with Helmholtz that the psychological origin and empirical application of mathematical geometry is to be explained by the structure of the group of rigid motions of the perceiver in perceptual space expressed in the condition of free mobility. Poincaré also entirely agrees that the structure of this group (i.e., the structure of perceptual space) is based on our motor or kinesthetic bodily sensations as we voluntarily move or displace our body in and through perceptual space. Accordingly, that aspect of perceptual space most relevant to geometry, for Poincaré, is what he calls motor space – the space generated by the group of bodily displacements. Finally, Poincaré also entirely agrees with Helmholtz that, precisely because the condition of free mobility leaves all three classical cases of geometries of constant curvature still open, specifically Euclidean geometry is neither a necessity of thought nor an a priori product of our form of spatial intuition. For Poincaré, however, an empirical explanation of the origin of Euclidean geometry is not the only remaining alternative. On the contrary, precisely because the exact mathematical concept of continuous group can only be an *idealization* of our rough and approximate experience of bodily displacements, the group we end up with must inevitably reflect our own free choice, which choice is guided but not constrained by the rough and approximate experience with which we begin.

For Poincaré, therefore, our choice of specifically Euclidean geometry (which he, like Helmholtz, has no reason to question) is based primarily on its mathematical simplicity: on the circumstance, namely, that only the Euclidean group of motions contains a normal subgroup of translations:

> Geometry is not an experimental science; experience forms merely the occasion for our reflecting upon the geometrical ideas which pre-exist in us. But the occasion is necessary; if it did not exist we should not reflect; and if our experiences were different, doubtless our reflections would be different. Space is not a form of our sensibility; it is an instrument which serves us not to represent things to ourselves, but to reason upon things.
>
> What we call geometry is nothing but the study of formal properties of a certain continuous group; so that we may say, space is a group. The notion of this continuous group exists in our mind prior to all experience; but the assertion is no less true of the notion of many other continuous groups; for example, that which corresponds to the geometry of Lobachevsky. There are, accordingly, several geometries possible, and it remains to be seen how a choice is made between them. Among the continuous mathematical groups which our mind can construct, we choose that which deviates least from that rough group, analogous to the physical continuum, which experience has brought to our knowledge as the group of displacements.
>
> Our choice is therefore not imposed by experience. It is simply guided by experience. But it remains free; we choose this geometry rather than that geometry, not because it is more *true*, but because it is more *convenient*.
>
> ... We choose the geometry of Euclid because it is the simplest.... [I]t is simpler because certain of its displacements are interchangeable with one another, which is not true of the corresponding displacements of the group of Lobachevsky.[35]

Poincaré's conventionalism is thus based, in the end, on the traditional Platonic gap between pure mathematical ideas and the rough perceptual experience from which they arise and to which they are to be applied. More precisely, when one combines this traditional gap with the Helmholtz–Lie solution to the "space-problem," one sees that there are three and only three mathematical counterparts to our intuitive experience of perceptual space – among which, therefore, a conventional choice must be made. By contrast, as we saw earlier, the entire point of Kant's own doctrine of *pure* spatial intuition is precisely to overcome this traditional Platonic gap – the gap, in Kantian terms, between reason and the understanding on one side and sensibility on the other. Indeed, as we argued at the end of §II, Kant's doctrine of the transcendental synthesis of the imagination is intended precisely to unite the understanding and sensibility once and for all, so that, in particular, "pure mathematics, *in its full precision*, [is made] applicable to objects of experience" (A165/B206, my emphasis).

Nevertheless, although Poincaré's conception of the role of spatial intuition in geometry is, in this respect, quite antithetical to the Kantian doctrine of pure spatial intuition, Poincaré's conception of the role of intuition in arithmetic is closely analogous to Kant's. As is well known, Poincaré vehemently opposes the logicist doctrine that arithmetic is a part of logic and hence a product, in Kantian terms, of the pure understanding. He holds instead that arithmetic is based on an irreducible intuition of succession or indefinite iteration, by which the mind is immediately aware of its own capacity indefinitely to repeat any given operation. It is this immediate awareness that grounds both the potential infinity of the number series and the characteristically mathematical form of reasoning expressed in mathematical induction. And this awareness is intuitive rather than conceptual precisely because neither fundamental property of our arithmetical thought is reducible, for Poincaré, to purely logical thinking – even if we widen our conception thereof to include the new mathematical logic. In this sense, Poincaré's defense of the idea that arithmetic is synthetic a priori is indeed genuinely Kantian, and Poincaré's conception of what we might call pure arithmetical intuition is in fact closely analogous to Kantian pure intuition. The one difference is that Poincaré's arithmetical intuition is not so directly and explicitly tied to sensibility, to the idea of time as the form of inner sense.[36]

What is perhaps not so well known is that Poincaré also emphasizes the importance of arithmetical intuition (the intuition of succession or indefinite iteration) in geometrical reasoning. As we have seen, Poincaré holds that the object of geometry is a group of rigid motions and thus, in accordance with the Helmholtz–Lie theorem, that space, although not necessarily Euclidean, must nonetheless have constant curvature. And it is precisely in this context that he appeals to arithmetical intuition:

[*S*]*pace is homogeneous and isotropic*. It may also be said that a movement which has once been produced may be repeated a second and a third time, and so on, without its properties varying. In the first chapter, where we discussed the nature of mathematical reasoning, we saw the importance which must be attributed to the possibility of repeating indefinitely the same operation. It is from this repetition that mathematical reasoning gets its power; it is, therefore, thanks to the law of homogeneity, that it has a hold on the geometric facts.[37]

This passage seems to be closely connected, in turn, with the circumstance that Poincaré, again in accordance with the Helmholtz–Lie theorem, explicitly excludes from consideration the more general theory of Riemannian manifolds including spaces of variable curvature:

If therefore the possibility of motion is admitted, there can be invented only a finite (and even a rather small) number of three-dimensional geometries. Yet this result seems contradicted by Riemann, for this savant constructs an infinity of different geometries, and that to which his name is ordinarily given is only a particular case. All depends, he says, on how the length of a curve is defined. Now, there is an infinity of ways of defining this length, and each of them may be the starting point of a new geometry.

That is perfectly true, but most of these definitions are incompatible with the motion of a rigid figure, which in the theorem of Lie is supposed possible. These geometries of Riemann, in many ways so interesting, could never therefore be other than purely analytic and would not lend themselves to demonstrations analogous to those of Euclid.[38]

Thus Poincaré appears to be perfectly clear (unlike Helmholtz, for example) that Riemann has indeed shown how to introduce the notion of distance or measurability into a manifold without relying on the motion of rigid bodies and hence on free mobility. Poincaré's claim is rather that nonhomogenous manifolds of variable curvature are not susceptible to Euclidean-style systems of demonstration, so that, therefore, they are not in the same sense synthetic.[39]

Now it is by no means clear precisely what Poincaré means by this assertion. Nevertheless, the way in which he juxtaposes geometrical and arithmetical intuition here may suggest a connection between the group of motions in a space of constant curvature and an iterative procedure of geometrical construction, a connection that would generalize what we argued earlier in the case of Kant's theory of geometrical construction and, specifically, Euclidean geometry.[40] In that case we argued that the Euclidean group of motions (the group of Euclidean translations and rotations) constitutes the basis, for Kant, of the procedure of construction with straight edge and compass underlying the proof structure of the *Elements*. This procedure iteratively generates the domain of Euclidean geometry so that, in particular, we are concerned here only with constructive existence claims and the potential infinite. We are concerned, that is, with a domain precisely analogous, in this respect, to the natural numbers. Moreover, as is well known, this feature of the domain of elementary Euclidean geometry can be expressed analytically by the circumstance that a Cartesian space over the entire field of real numbers is by no means necessary for representing the existence claims implicit in Euclid's postulates. On the contrary, the domain of elementary Euclidean geometry is represented precisely by a Cartesian space over the much smaller Euclidean subfield of the reals, which results by closing the rationals under the operation of extracting real square roots. (And it is this representation, of course, that we use to prove the impossibility within elementary Euclidean geometry of various "higher" constructions such as the trisection of an angle.)

It is interesting, then, that something closely analogous is true in all three classical cases of spaces of constant curvature. In all three cases, one can formulate an elementary geometry where, in place of the Dedekind continuity axiom, one simply has an axiom of intersection for straight lines and circles. The domains of these elementary geometries therefore consist of precisely those points generated by straight-edge and compass constructions in the sense of each of the geometries in question, and each of these domains is analytically representable by an appropriate space over a Euclidean subfield of the reals – the familiar Cartesian space in the case of Euclidean geometry, certain Klein spaces in the cases of hyperbolic and elliptic geometry. Moreover, in this more general analytic treatment, all three cases are viewed in accordance with the Cayley–Klein program as embedded within projective geometry (in the elliptic case, the embedding is trivial), so that, in particular, the three different groups of motions appear as subgroups of the projective group. The analytic representations in question are then induced by corresponding analytic representations of projective geometry. In this sense, there does indeed seem to be a general connection between groups of rigid motions and geometrical construction subsisting in all three classical cases of constant curvature.

The most interesting case of this situation occurs in hyperbolic or Bolyai–Lobachevsky geometry, a central feature of which is the existence of limiting or asymptotic parallel lines.[41] It is not only the case that, given a line l and a point P not on l, there are an infinity of lines through P that do not intersect l (and are in this sense parallel to l), but among all such nonintersecting lines, there are exactly two distinguished ones, the limiting or asymptotic parallels to l through P, that precisely divide the set of all lines through P into two classes – those that lie within the angle determined by the two asymptotic parallels on the same side as l (i.e., within the region $\mathcal{R}$ in Figure 1) and intersect l, and those that do not lie within this angle and do not intersect l.

The pair of asymptotic parallel lines through P therefore determines a Dedekind cut in the set of all lines through P (more precisely, in the set of all rays or half-lines originating at P) with respect to the property of intersecting line l, and, accordingly, the existence of such lines is traditionally justified by a Dedekind continuity axiom. It was Hilbert in 1903 who showed that one could,

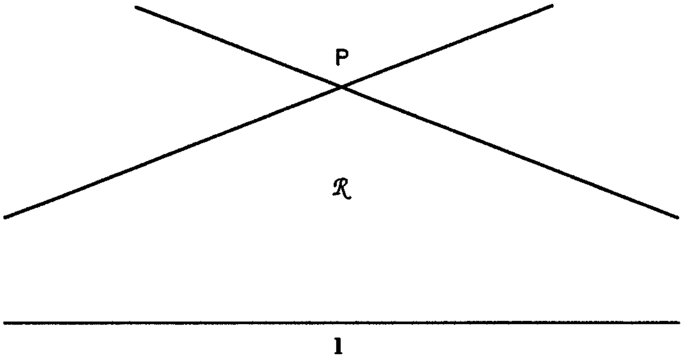

Figure 1

instead, simply add an axiom asserting the existence of asymptotic parallels to his axioms of incidence, order, and congruence characterizing absolute geometry, and one thereby obtains all the usual theorems of hyperbolic geometry without needing to invoke full Dedekind continuity.[42] The key part of Hilbert's approach is the construction of a field based on equivalence classes of asymptotic parallels (the so-called end calculus), which then can be used to coordinatize the set of points so as, in effect, to embed the geometry in question within projective geometry. From this embedding within projective geometry, the usual formulas of hyperbolic geometry then follow. It turns out that the field thus constructed by Hilbert is precisely a Euclidean field. In particular, the axiom of intersection for lines and circles is a consequence of Hilbert's axiom of asymptotic parallels.

It is natural to ask whether the converse also holds. That is, given Hilbert's axioms of incidence, order, and congruence characterizing absolute geometry, the negation of the Euclidean parallel postulate, and the axiom of intersection for lines and circles, can we then derive the existence of asymptotic parallels? If so, we would, in effect, have constructed the asymptotic parallels with straight edge and compass within hyperbolic geometry. And it is in fact the case that Bolyai himself gave a construction with straight edge and compass of the asymptotic parallels already in 1832[43] (where, in Figure 2, *PX* and *PY* are congruent to *QR*).

It turns out, however, that to complete the derivation in question, one also needs to invoke the Archimedean axiom. This follows from work of Hessenberg, Hjelmslev, and Bachmann, which yields an embedding of any geometry satisfying the axioms of absolute geometry into projective geometry so as thereby to construct a canonical coordinatization by a field K. Moreover, the axiom of intersection for lines and circles holds if and only if K is Euclidean. But, if the

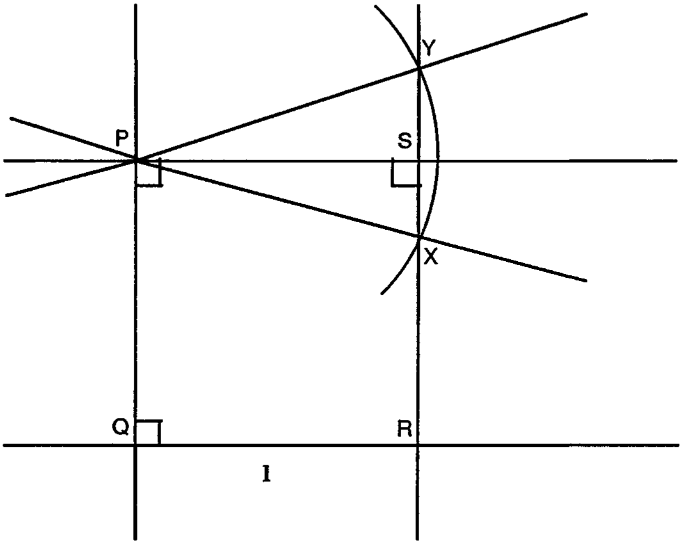

Figure 2

negation of the Euclidean parallel postulate holds and K is non-Archimedean, then Bolyai's construction yields lines through a point P having a "common perpendicular at infinity" with the given line l, which lines, however, are not necessarily asymptotically parallel to l. Each such line through P meets l in an ideal point at infinity, but the ideal points at infinity do not necessarily have a minimum or limiting value. If K is Archimedean, by contrast, then Bolyai's construction does yield asymptotic parallels, and, in fact, the resulting geometry must be isomorphic to the usual Klein model for hyperbolic geometry over an Archimedean Euclidean field (where, in Figure 3, the points in our model are all interior to the bounding circle of ideal points, the polar constructions outside the bounding circle depict the perpendicularity relations of Figure 2, and the interior circle with center P and radius equal to QR appears as a conic). (Without the Archimedean axiom, by contrast, we can construct models in which the interior points are all infinitesimal, the bounding "circle" of ideal points consists of all finite points, and the ultra-ideal points outside the bounding circle are infinite; Bolyai's construction then yields ideal points on the bounding circle, but there is no limiting or minimum value.) In this sense, by taking the Archimedean axiom as an additional constructive constraint, we can give a constructive treatment of Bolyai–Lobachevsky geometry analogous to Euclidean geometry.

As noted earlier, an analogous treatment can be given of elliptic geometry, although here, we of course need to generalize the axioms of absolute geometry as well. In all three classical cases of constant curvature, the connection we

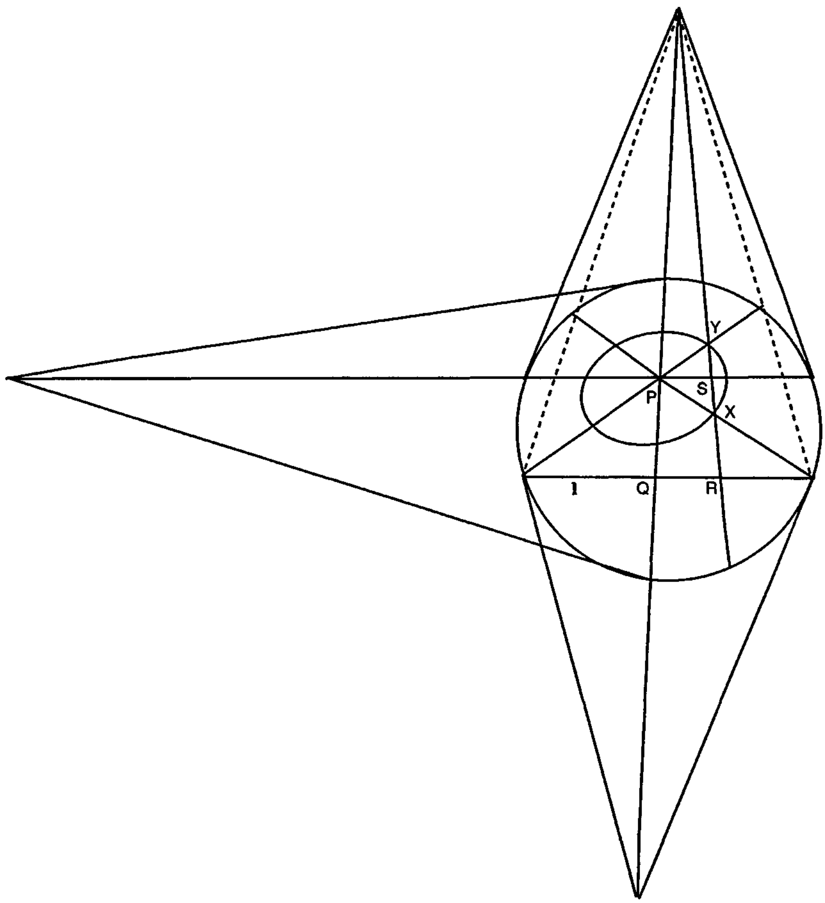

Figure 3

have attributed to Kant between space as a form of intuition or outer perception given by a group of motions and space as an object of geometrical construction therefore appears to hold.[44] Yet it is also interesting to note, finally, that if we do not insist on such a connection with geometrical construction, it is still possible to envision a "phenomenological" foundation, based on a consideration of the possible motions of the perceiving subject in and through perceptual space, of the more general class of Riemannian manifolds, including spaces of variable curvature. This, in fact, was precisely the philosophical motivation behind Hermann Weyl's generalization of the Helmholtz–Lie solution to the "space-problem." Weyl developed his analysis in the context of Husserlian phenomenology and, in particular, as a correction to Oskar Becker's (Husserl's student) phenomenological justification of specifically Euclidean geometry.[45] Becker began by considering the phenomenological subject as embedded in space at a point of view with respect to which it can change both its orientation and its position. By imposing a condition of free mobility on such possible changes, we then derive the constant curvature of the given space, and by further requiring that the translations constitute a normal subgroup, we arrive at specifically Euclidean geometry. For Weyl, by contrast, we do not assume that Helmholtzian free mobility is possible and thus that the perceptual space of the phenomenological subject must have constant curvature. Instead, Weyl begins with the idea of an infinitesimal rotation group at every point and then fixes the associated metric as Pythagorean or infinitesimally Euclidean by requiring that an affine connection – and thus the idea of infinitesimal translation from the initial phenomenological point of view – be thereby determined *uniquely*:

> A way for understanding the Pythagorean nature of the metric expressed in the Euclidean rotation group precisely on the basis of the separation of a priori and a posteriori has been given by the author: Only in the case of this group does the intrinsically accidental quantitative distribution of the metric field uniquely determine in all circumstances (however it may have been formed in the context of its a priori fixed nature) the infinitesimal parallel displacement: the non-rotational progression from a point into the world. This assertion involves a deep mathematical theorem of group theory which I have proved. I believe that this solution of the space-problem plays the same role in the context of the Riemann-Einstein theory that the Helmholtz-Lie solution (section 14) plays for rigid Euclidean space. Perhaps the postulate of the unique determination of "straight-progression" can be also justified from the requirements of the phenomenological constitution of space; Becker would still like to ground the significance of the Euclidean rotation group for intuitive space on Helmholtz's postulate of free mobility.[46]

Although Weyl's group-theoretical solution to the generalized "space-problem" of course retains its mathematical interest entirely independently of this connection with Husserlian phenomenology, the philosophical motivations behind Weyl's approach attest, nevertheless, to the enduring fascination of the idea that geometry is to be based on a consideration of space as a (kinematical) form of intuition.[47]

In Weyl's work, therefore, the program, begun by Kant, of considering the geometry of space as given in our form of outer intuition has in a sense come full circle. In Kant, as we have seen, this program has both a logical or constructive and a phenomenological or perceptual dimension. The logical or constructive side of Kant's conception is grounded in the proof procedure of Euclid's *Elements*, where we iteratively generate the objects of geometry by a definite procedure of construction, so that the objects in question constitute a *potentially* infinite totality. The phenomenological or perceptual side of Kant's conception is expressed by the "action of the understanding on sensibility" (B152), whereby the subject imaginatively locates itself in space at a definite point of view and with a definite orientation, so that, by virtue of the resulting formal structure of pure outer intuition, the space in which we perceive outer objects is necessarily both singular or unitary and infinite or unbounded. In Euclidean geometry, moreover, we find mathematical counterparts to both sides of Kant's conception, in that the translations and rotations at the basis of the Euclidean group of rigid motions generate the two fundamental elements of Euclidean construction – straight lines and circles – as their orbits. In the nineteenth century, Kant's conception is both generalized and radically transformed in the work of Helmholtz and Poincaré, where a group-theoretical and perceptual/kinematical interpretation of the foundations of geometry is extended also to the classical non-Euclidean geometries of constant curvature. Since, however, this nineteenth-century generalization now has an entirely conceptual model of geometry given by Riemann's theory of manifolds, spatial perception as it figures in the foundational conceptions of Helmholtz and Poincaré is now, in Kantian terminology, empirical as opposed to pure intuition. Nevertheless, it is still possible, from a mathematical point of view, to connect this generalized group-theoretical conception of geometry with an appropriate generalization of Euclidean construction. In the case of the variably curved Riemannian manifolds employed in the general theory of relativity, by contrast, the approach taken by Helmholtz and Poincaré must itself be radically transformed. And the import of Weyl's reaction to this situation, from the present point of view, is that a group-theoretical and perceptual/kinematical interpretation of the foundations of geometry can, in a new sense, be sustained, whereas, at the same time, the connection with a definite procedure of construction is abandoned. In Weyl's work, we might say that we find a definitive divorce between the logical or constructive and the phenomenological or perceptual dimensions of Kant's original doctrine.

NOTES

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1. M. Friedman, "Kant's Theory of Geometry," *Philosophical Review*, 94 (1985): 455–506, reprinted, with revisions, as Chapter 1 of *Kant and the Exact Sciences*

(Cambridge, MA: Harvard U. Press, 1992); C. Parsons, "The Transcendental Aesthetic," in P. Guyer (ed.), *The Cambridge Companion to Kant* (Cambridge UK: Cambridge U. Press, 1992). Parsons' original discussion of this issue, to which I was responding in 1985, is "Kant's Philosophy of Arithmetic," in S. Morgenbesser, P. Suppes, and M. White (eds.), *Philosophy, Science, and Method: Essays in Honor of Ernest Nagel* (New York: Cornell U. Press, 1969), 568–94, reprinted, with a postscript, in *Mathematics in Philosophy* (Ithaca, NY: Cornell University Press, 1983), 110–49.

2. E. Beth, "Über Lockes 'Allgemeines Dreieck'," *Kant-Studien* 49 (1956–7): 361–80; J. Hintikka, "On Kant's Notion of Intuition (Anschauung)," in T. Penelhum and J. MacIntosh (eds.), *Kant's First Critique* (Belmont, Calif.: Wadsworth, 1969); "Kant's 'New Method of Thought' and His Theory of Mathematics," *Ajatus* 27 (1965): 37–43; "Kant on the Mathematical Method," *The Monist*, 51 (1967): 352–75. These last two are reprinted in *Knowledge and the Known* (Dordrecht: Reidel, 1974).
3. "The Transcendental Aesthetic," p. 66; this refers back to "Kant's Philosophy of Arithmetic," p. 112 of *Mathematics in Philosophy.*
4. E. Carson, "Kant on Intuition in Mathematics," presented to the Canadian Philosophical Association in the spring of 1994. This paper has appeared more recently as "Kant on Intuition in Geometry," *Canadian Journel of Philosophy*, 27 (1997): 489–512.
5. See Parsons, "The Transcendental Aesthetic," p. 70: "[T]here is a phenomenological fact to which [Kant] is appealing: places, and thereby objects in space, are given in a [single] space, therefore with a 'horizon' of surrounding space."
6. See Parsons, "The Transcendental Aesthetic," pp. 77–8: "[Euclidean constructions] are constructions *in* intuition; space is, one might say, the field in which the constructions are carried out; it is by virtue of the nature of space that they *can* be carried out."
7. All translations from Kant's German are my own and are cited, except for the *Critique of Pure Reason*, by volume and page numbers of the Akademie edition of *Kant's gesammelte Schriften* (Berlin: Reimer (later Walter de Gruyter), 1902–); the *Critique of Pure Reason* is cited by the standard A and B pagination of the first (1781) and second (1787) editions, respectively. Kant's reply to Eberhard, *On a Discovery According to Which Any New Critique of Pure Reason Has Been Made Superfluous by an Earlier One* (1790), is translated, together with valuable supplementary materials, in H. Allison (ed.), *The Kant-Eberhard Controversy* (Baltimore: Johns Hopkins U. Press, 1973): the present passage can be found on pp. 175–6.
8. I thus disagree with Henry Allison's contention, in *Kant's Transcendental Idealism* (New Haven: Yale U. Press, 1983), pp. 83–6, that "outside [*außer*]" has a nonspatial meaning here – meaning "distinct from me (the self)" in the first clause and "(numerically) distinct from one another" in the second clause – so that, in the second clause, Kant is referring to space's role as a principle of individuation. It seems to me, on the contrary, that the parenthetical insertion in the first clause, as well as the "and next to" added (in the second edition) to the second clause, make it clear that "outside" has a spatial meaning. Kant is not outlining an abstract principle of individuation here but rather articulating an a priori perceptual structure: "outside me" is perceptually indexical, and has the force of "outside of (and thus capable of being seen from) this point of view."
9. Compare the above-quoted passage from B154–5 on "representing the three dimensions of space" with the discussion of orientation in *On the First Ground*

of the Distinction of Regions in Space (1768) at Ak. 2, pp. 378–9. This latter discussion begins with the construction, "because of the three dimensions [of space]," of "three surfaces . . . that all intersect one another at right angles."

10. Allison's main ground for rejecting a spatial reading of "outside" in the passage at A23/B38 (note 8, above) is that this makes Kant's claim appear tautological and thus analytic. However, whereas it is indeed tautological that all "outer" objects are in space, it does not follow that the articulation of the a priori structure of this perceptual space is itself analytic. On the contrary, this formal structure is described precisely by the *synthetic* a priori science of geometry; and the task of the transcendental philosopher is then to describe the human cognitive faculties that make this possible. In the end, therefore, the difference between the two readings rests, I believe, on Allison's attempt (which, in fact, is common to most interpretations of the Transcendental Aesthetic) to make the argument of the Metaphysical Exposition entirely independent of the consideration of geometry (which is then supposed to be confined to the Transcendental Exposition): see *Kant's Transcendental Idealism*, pp. 81–2, 98–9. I am indebted to Graciela De Pierris for discussion of this point.
11. The priority of motion and the circumstance that construction is essentially kinematical rather than instantaneous implies that *points* are not independently constructible – they emerge only as products of the process of drawing lines and describing circles (as intersections, endpoints, and so on). This clarifies Kant's claim, made in the context of a discussion of continuous, "flowing" quantities, that "[p]oints and instants are only limits, that is, mere places of their [space's or time's] limitation; but places always presuppose those intuitions that they limit or are to determine" (A169/B211).
12. How do we know, in particular, that Euclidean constructions can indeed be carried out – and also iterated indefinitely? On the logical interpretation, since there is no purely logical or conceptual representation possible, the only way we can even think of or represent, say, the proposition that a circle is always constructible with a given center and radius, is by actually *possessing* the construction in question (as a Skolem function, as it were, for the existential quantifier); and, if we have the construction, the proposition is then automatically true. The proposition is thus a priori true, because its truth is a condition of its mere possibility. On the phenomenological interpretation, by contrast, the truth of such geometrical axioms is not already settled by their mere possibility: geometrical intuition and perceptual spatial "facts" are then called in to settle this question. See *Kant and the Exact Sciences*, Ch. 1, p. 66, and especially Ch. 2, §IV. (This note and the preceding one were prompted by queries from Michael Dickson.)
13. Parsons has suggested to me in conversation that an emphasis on the importance of directly perceptible phenomenological facts in grounding or explaining the axioms of geometry need not involve a commitment to an epistemological foundation for geometry given from outside this science itself; on the contrary, it may simply involve the attempt to articulate the significance of intuitively spatial evidence *within* the science of geometry. This is an important suggestion, which I will touch upon briefly later.
14. Eberhard's appeal to Apollonius appears in his *Philosophisches Magazin*, I (1789): 158–9. See Ak. 20, p. 505. Kant orchestrated a full-scale counterattack, which included (besides his own *On a Discovery*) a review essay by Reinhold (Ak. 20, pp. 385ff). See Allison's *The Kant-Eberhard Controversy*, for further details and a selection of relevant materials.

15. See T. L. Heath (ed.), *Apollonius of Perga: Treatise on Conic Sections* (Cambridge UK: Cambridge U. Press, 1896), pp. 1–9 (Prop. I.11 is Proposition 1 in Heath's numbering). The notation "$y^2 = ax$" is used by Kant.
16. This terminology derives from Pappus's commentary on Apollonius's two books on plane loci. Loci (curves) that are neither plane nor solid – including both higher algebraic curves and transcendental curves – are here termed "curvilinear." See A. Jones (ed.), *Pappus of Alexandria: Book 7 of the Collection* (Berlin: Springer, 1986), pp. 104–7.
17. Eberhard explains in a correction published in his *Philosophisches Magazin,* III (1790/91): 205–7, that he had mistakenly cited Borelli's 1661 edition of Books V–VII; the edition in question is actually that of Books I–IV by Claudius Richardus (1655).
18. In the *Metaphysical Foundations of Natural Science* (1786), Kant distinguishes between "geometrical" and "mechanical construction" (Ak. 4, 493) and, accordingly, between the "*mathematical construction*" and the "*mechanical* execution [*Ausführung*]" of the composition of velocities – where the latter shows "how it can be *brought forth* through nature or art by means of certain instruments and forces" (494).
19. Descartes's new version of the distinction between "geometrical" and "mechanical" curves, which is explicitly intended as a correction to the traditional classification of Pappus (see note 16), occurs at the beginning of Book II of the *Géometrié*, in a section entitled "What curved lines are admitted in geometry" – see D. Smith and M. Latham (trans.), *The Geometry of René Descartes* (La Salle, Il: Open Court, 1925), pp. 40–9. For discussion, see A. Holland, "Shifting the Foundations: Descartes's Transformation of Ancient Geometry," *Historia Mathematica*, 3 (1976): 21–49, and especially H. Bos, "On the Representation of Curves in Descartes' *Géométrie*," *Archive for History of Exact Science*, 24 (1981): 295–338 and "The Structure of Descartes' *Géométrie*," in G. Belgioiso (ed.), *Descartes: il Metodo e i Saggi* (Rome, Istituto della Enciclopedia Italiana, 1990). I am indebted to Bos, and also to Mark Wilson, for urging me to consider this question of the scope of geometrical construction.
20. Newton gives a well-known mechanical construction of conics (which may very well have been familiar to Kant), by rotating lines and thereby describing intersections, in Lemma XXI of *Principia*, Book I.
21. As noted earlier, I therefore reject the idea – characteristic of the phenomenological interpretation (compare note 5) – that the unity and unboundedness of space can be directly and immediately given as some kind of quasi-perceptual fact.
22. Compare Kant's remarks about the relationship between the "category of *magnitude*" and the perception of the spatial figure of a house at B162. I am indebted to Parsons for prompting this discussion of the connections among intuitive spatial unity, the transcendental unity of apperception, and the categories of quantity.
23. These latter questions are central to Kant's *Metaphysical Foundations of Natural Science*. See Chapters 3 and 4 of *Kant and the Exact Sciences*, as well as my "Causal Laws and the Foundations of Natural Science," in P. Guyer (ed.), *The Cambridge Companion to Kant*, pp. 161–99.
24. From a modern, four-dimensional point of view, we are concerned only with continuous transformations within a single plane of simultaneity, and thus the dynamical question of how different planes of simultaneity are related to one another over time is not relevant here. Kant himself, in the first chapter on Phoronomy of the *Metaphysical Foundations of Natural Science*, says that, in phoronomy, "motion can be considered solely as *describing of a space*, but still in such a way that I attend not

merely, as in geometry, to the space that is described, but also to the time in which and thus the speed with which a point describes the space" (Ak. 4, p. 489). And in §24 of the second edition Deduction, in the passage from B154–5 cited earlier, after giving examples of geometrical synthesis, Kant describes how we represent "time itself" by attending, not to the space described in drawing a straight line, but to the act of successive determination by which we thereby determine *inner* sense and thus "first produce the concept of succession." (The concept of succession is of course a component of the *dynamical* category of causality – more precisely, of its schema: A144/B183 and compare B291–2.) I am indebted to Howard Stein for prompting this discussion of the nature and status of geometrical motion.

25. The basic idea of using Helmholtz's kinematical conception of spatial intuition to build a bridge between the phenomenological and logical approaches to interpreting Kant's own theory of spatial intuition is due to Robert DiSalle – in comments on the paper by Emily Carson cited in note 4. My own work on the present paper grew directly out of conversations with DiSalle.
26. "Über die Tatsachen, die der Geometrie zugrunde liegen," *Nachrichten von der königlichen Gesellschaft der Wissesschaften zu Göttingen*: no. 9, (1868): 39–71, translation in R. Cohen and Y. Elkana (eds.), *Hermann von Helmholtz: Epistemological Writings* (Dordrecht, The Netherlands: Reidel, 1977), Ch. II.
27. "Über die Ursprung und die Bedeutung der geometrischen Axiomen," first given as a lecture in 1870 and published in *Populäre wissenschaftliche Vorträge*, Vol. 2 (Braunschweig: F. Vieweg, 1871). I cite from H. Hörz and S. Wollgast (eds.), *Philosophische Vorträge und Aufsätze* (Berlin: Akademie Verlag, 1971), p. 214 – this corresponds to the translation in R. Cohen and Y. Elkana (eds.), *Hermann von Helmholtz: Epistemological Writings*, p. 23.
28. Appendix III to the address of 1878, "Die Tatschen in der Wahrnemung," first published in *Vorträge und Reden* (Braunschweig: F. Vieweg, 1884). I cite from H. Hörz and S. Wollgast (eds.), *Philosophische Vorträge und Aufsätze*, p. 299, which corresponds to Cohen and Elkana (eds.), *Hermann von Helmholtz*, pp. 162–3. "Space can be transcendental without the axioms being so" is the title of Appendix II.
29. I thus interpret Helmholtz as holding that the general, "transcendental" form of spatial intuition includes the condition of free mobility (and thus constant curvature) but not the properties of specifically Euclidean geometry (zero curvature). As Howard Stein, in particular, has emphasized to me, this is certainly not the only possible interpretation: one might also take the most general, "transcendental" form of space to include, for example, only topological and manifold properties. The present reading is supported by the following passage from "On the Origin and Meaning of the Axioms of Geometry," which initiates the criticism of the specifically Kantian theory of intuition:

    We will now have to ask further where those particular determinations come from that characterize our space as plane space, since these, as has been shown, are not included in the general concept of an extended magnitude of three dimensions and free mobility of the structures contained therein. They are not *necessities of thought*, which flow from the concept of such a manifold and its measurability or from the most general concept of a rigid structure contained therein and its freest mobility. (Hörz and Wollgast (eds.), *Philosophische Vorträge*, pp. 206–7; Cohen and Elkana (eds.), *Hermann von Helmholtz*, p. 17)

    One should also note that whenever Helmholtz explicitly states what he calls "the axioms of geometry," these always characterize specifically Euclidean space.

30. See "The Facts in Perception" – Hörz and Wollgast (eds.), *Philosophische Vorträge*, pp. 262–5; Cohen and Elkana (eds.), *Hermann von Helmholtz*, pp. 128–31.
31. See Hörz and Wollgast (eds.), *Philosophische Vorträge*, pp. 256–8; Cohen and Elkana (eds.), *Hermann von Helmholtz* pp. 122–4.
32. Kant's discussion of orientation cited in note 9 uses the human body to determine the relations above–below, right–left, and forward–backward. This amounts to using one's body to pick out a particular triad of mutually perpendicular line segments corresponding, respectively, to these three relations. This procedure should be viewed, I believe, as an account of how we *apply in experience* the purely geometrical notion of orientation – which notion is itself given by an entirely arbitrary construction of three mutually perpendicular line segments in *pure* intuition. It would be interesting to apply these ideas to Kant's conception of incongruent counterparts, but this will have to wait for another occasion. (This note was prompted by comments and suggestions from Robert Hanna, Onora O'Neill, and Allen Wood.)
33. This is connected with the circumstance, remarked in note 24, that the purely geometrical motion involved in figurative synthesis does not yet involve the questions of *time*-determination characteristic of the dynamical as opposed to the mathematical categories. Compare A160/B199:

    > In the application of the pure concepts of the understanding to possible experience the use of their synthesis is either *mathematical* or *dynamical*. For it applies partly to mere *intuition*, partly to the *existence* of an appearance in general. But the a priori conditions of intuition are necessary throughout in relation to a possible experience, those of the existence of objects of a possible empirical intuition are in themselves only contingent.

    The dynamical categories essentially involve the conditions for transforming perceptions or empirical intuitions into law-governed experience, and thus, as I have argued elsewhere (see note 23), they also involve the conditions for transforming apparent motions into true motions – where the (Newtonian) laws of motion here realize or embody the Analogies of Experience. The purely mathematical synthesis expressed in pure geometrical motion, by contrast, has nothing to do with the laws of motion. Once we follow Helmholtz in basing geometry on real bodily motion, however, we simply cannot avoid entangling geometry with the laws of motion: we cannot avoid facing the fact, in modern terms, that the four-dimensional structure of space-time is primary.
34. The crucial point is that, on the phenomenological interpretation, the truths of geometry appear as brute "perceptual facts" – even if they are conceived as intuitively evident truths internal to the (supposedly) a priori science of geometry as in note 13. On this kind of interpretation, there is no particular difficulty in thinking or conceiving the truths of geometry independently of spatial intuition, and the latter is then called in only to establish that some particular set of axioms (the Euclidean axioms) is in fact true. On the logical interpretation, by contrast, the truths of geometry function as a priori preconditions without which it would be impossible even to think of or to conceive spatial structures in the first place: without the truths of geometry there would simply be no "spatial facts" (see note 12). If one now asks where the "spatial facts" invoked by the phenomenological interpretation come from, this interpretation is then vulnerable, from a philosophical point of view, to naturalism and empiricism – according to which such "facts" must rest, in the end, on contingent conditions of our perceptual apparatus and/or contingent

characteristics of physical space (see Parsons, "The Transcendental Aesthetic," pp. 72–5). Here I am again indebted to De Pierris – compare her discussion of Parsons's paper in her "Review of *The Cambridge Companion to Kant*," *Ethics*, 104 (1994): 655–7.

35. "On the Foundations of Geometry," *The Monist*, 9 (1898), pp. 41–3.
36. For a discussion of Poincaré's theory of arithmetic, see J. Folina, *Poincaré and the Philosophy of Mathematics* (New York: Macmillan, 1992).
37. *La Science et l'Hypothèse* (Paris: Flammarion, 1902), Ch. IV – I cite from the translation of G. Halsted in *The Foundations of Science* (Lancaster, Pa.,: The Science Press, 1913), p. 75.
38. *La Science et l'Hypothèse*, Ch. III, citing from G. Halsted (trans.), *Foundations of Science*, p. 63. In the popular Dover edition of *Science and Hypothesis* (New York, 1952), p. 48, the last sentence is incorrectly translated as: "These geometries of Riemann, so interesting on various grounds, can never be, therefore, purely analytical, and would not lend themselves to proofs analogous to those of Euclid" – thereby entirely reversing its sense (and the preceding sentence on p. 47 incorrectly has "variable figure" instead of "invariable figure").
39. Since Poincaré thus rules out nonhomogeneous manifolds of variable curvature, his conventionalism is entirely incompatible with the general theory of relativity – a circumstance that has led to considerable confusion among his followers: see my "Poincaré's Conventionalism and the Logical Positivists," in J.-L. Greffe, G. Heinzmann, and K. Lorenz (eds.), *Henri Poincaré: Science and Philosophy* (Berlin and Paris: Blanchard and Akademie Verlag, 1996), pp. 333–44. Indeed, since Poincaré (like Helmholtz) bases geometry on the free mobility of real physical bodies, his conception is also incompatible with the space-time structure of special relativity – where, despite the fact that each individual plane of simultaneity is Euclidean, there is still no free mobility of rigid bodies in space-time (compare notes 24 and 33).
40. Poincaré's explicit discussion of "The Reasoning of Euclid" in "On the Foundations of Geometry," pp. 32–4, focuses on proofs that proceed by translating and rotating figures rather than on Euclidean constructions. Yet, as we will see, it is nonetheless possible to forge a connection between these two ideas. Helmholtz suggests such a connection in "On the Origin and Meaning of the Axioms of Geometry" – Hörz and Wollgast (eds.), *Philosophische Vorträge*, pp. 190–1; Cohen and Elkana (eds.), *Hermann von Helmholtz*, pp. 4–5.
41. For this and the next three paragraphs, see M. Greenberg, "Euclidean and Non-Euclidean Geometries Without Continuity," *American Mathematical Monthly*, 86: 757–64.
42. "Neue Begründung der Bolyai–Lobatschefsckyschen Geometrie," *Mathematische Annalen*, 57 (1903): 137–50. This appears as Appendix III to *Foundations of Geometry* (La Salle: Open Court, 1971).
43. Section 34 of J. Bolyai, *Scientiam Spatii Absolute Veram*, published as an Appendix to W. Bolyai, *Tentamen Juventutem Studiosam in Elementa Matheseos Purae*, translated in G. Bonola, *Non-Euclidean Geometry* (New York: Dover, 1955), pp. 37–8.
44. For the elliptic case, see W. Schwabhäuser, "On Models of Elementary Elliptic Geometry," in J. Addison (ed.), *The Theory of Models* (Amsterdam: North-Holland, 1965). If one wants an elementary geometry going beyond straight-edge and compass constructions to include all algebraic curves in the manner of Descartes (note 19), then one can, by using a first-order continuity schema, also construct

geometries over real closed fields in all three cases of constant curvature: see W. Schwabhäuser, "Metamathematical Methods in Foundations of Geometry," in Y. Bar-Hillel (ed.), *Logic, Methodology and Philosophy of Science* (Amsterdam: North-Holland, 1965).

45. Weyl explains the background of his mathematical investigations in Husserlian phenomenology in the Introduction to *Raum, Zeit, Materie* (Berlin: Springer, 1918), translated as *Space-Time-Matter* (New York: Macmillan, 1952). For Becker, see "Beiträge zur phänomenologischen Begründung der Geometrie und ihrer physikalishen Anwendung," *Jahrbuch für Philosophie und phänomenologische Forschung*, 6 (1923): 385–560. For further discussion, in the context of Rudolf Carnap's contrasting (nonkinematical) conception of geometrical intuition developed in his dissertation of 1921, see my "Carnap and Weyl on the Foundations of Geometry and Relativity Theory," *Erkenntnis*, 42 (1995): 247–60.
46. *Philosophie der Mathematik und Naturwissenschaft* (Berlin: Leibniz Verlag, 1927), §18, pp. 99–100, translated as *Philosophy of Mathematics and Natural Science* (Princeton, NJ: Princeton University Press, 1949), p. 137.
47. In "Die Einzigartigkeit der Pythagoreischen Maßbestimmung," *Mathematische Zeitschrift*, 12 (1922): 114–46, where Weyl first proves his group-theoretical theorem, he begins by stating that the infinitesimally Euclidean nature of the metric is "*characteristic of space as form of appearance*" (p. 116). (One should note that Weyl's "purely infinitesimal" approach involves an extended conception of metrical structure as well, where a "Weyl structure" on a manifold consists of a *class* of conformally equivalent Riemannian metrics, each paired with an accompanying "gauge factor.")